MTH 161 - Lecture 8

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2.2 : The Derivative as a function

Last class, we consider the of a function f at a fixed number a

$$\boxed{f'(a) = \lim_{h \to 0} \frac{f(a+h) - f(a)}{h}}$$

So next, what we would like to do is let the number a vary.

If we replace a by x, we obtain

$$\boxed{f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}}$$

So given a number x for which the limit exists, we assign to x the number $f'(x)$

So we can regard f' as a new function, called the derivative of f.

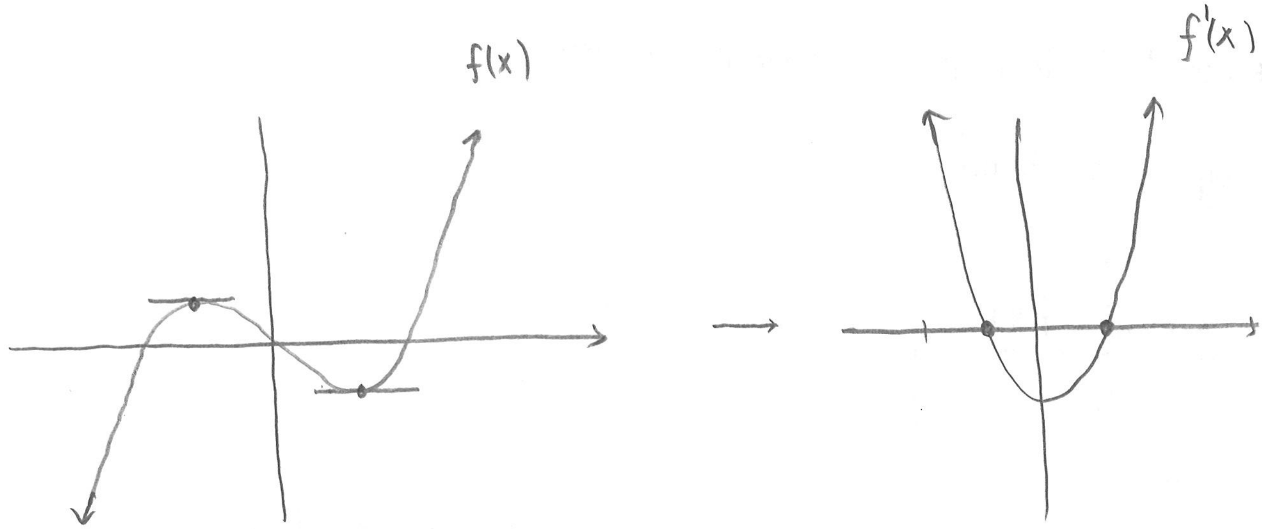
Ex If $f(x) = x^3 - x$, find a formula for $f'(x)$

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{([x+h]^3 - [x+h]) - (x^3 - x)}{h}$$

$$= \lim_{h \to 0} \frac{x^3 + 3x^2h + 3xh^2 + h^3 - x - h - x^3 + x}{h}$$

$$= \lim_{h \to 0} \frac{3x^2h + 3xh^2 + h^3 - h}{h}$$

$$= \lim_{h \to 0} \frac{h(3x^2 + 3xh + h^2 - h)}{h} = \lim_{h \to 0} (3x^2 + 3xh + h^2 - h) = 3x^2 - 1$$



<u>Ex</u> If $f(x) = \sqrt{x}$, find the derivative of f. State the domain of f'.

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h} = \lim_{h \to 0} \frac{\sqrt{x+h} - \sqrt{x}}{h}$$

$$= \lim_{h \to 0} \left(\frac{\sqrt{x+h} - \sqrt{x}}{h} \cdot \frac{\sqrt{x+h} + \sqrt{x}}{\sqrt{x+h} + \sqrt{x}} \right) = \lim_{h \to 0} \frac{(x+h) - x}{h(\sqrt{x+h} + \sqrt{x})}$$

$$= \lim_{h \to 0} \frac{\cancel{h}}{\cancel{h}(\sqrt{x+h}+\sqrt{x})} = \lim_{h \to 0} \frac{1}{\sqrt{x+h}+\sqrt{x}}$$

$$= \frac{1}{\sqrt{x}+\sqrt{x}} = \frac{1}{2\sqrt{x}}$$

Domain of f' : $[0,\infty) \cap x \neq 0 \Rightarrow (0,\infty)$

OTHER NOTATIONS

$y = f(x)$ to indicate that the independent variable is x and the dependent variable is y, the other notations for derivatives are

$$f'(x) = y' = \frac{dy}{dx} = \frac{df}{dx} = \frac{d}{dx}(f(x))$$

$\frac{dy}{dx}$ is not really a ratio but it is just another way of writing $f'(x)$.

To write $f'(a)$, we write $\left.\frac{dy}{dx}\right|_{x=a}$.

Differentiable functions

A function is differentiable at a if $f'(a)$ exists.

It is differentiable on an open interval (a,b) [or (a,∞) or $(-\infty,a)$ or $(-\infty,\infty)$]

if it is differentiable at every number in the interval.

Ex Where is the function $f(x) = |x|$ differentiable?

Sol^n For $x > 0$, then $|x| = x$, and hence we can choose h small enough that $x+h > 0$, and hence $|x+h| = x+h$. Therefore, for $x > 0$, we have

$$f'(x) = \lim_{h\to 0} \frac{|x+h| - |x|}{h} = \lim_{h\to 0} \frac{x+h-x}{h} = \lim_{h\to 0} \frac{h}{h} = \lim_{h\to 0} 1 = 1$$

so f is differentiable for any $x > 0$.

Similarly, for $x < 0$ we have that $|x| = -x$ and h can be small enough so that $x+h < 0$, and hence $|x+h| = -(x+h)$. Therefore, for $x < 0$

$$f'(x) = \lim_{h\to 0} \frac{|x+h| - |x|}{h} = \lim_{h\to 0} \frac{-(x+h) - (-x)}{h} = \lim_{h\to 0} \frac{-h}{h} = \lim_{h\to 0} -1 = -1$$

so f is differentiable for any $x < 0$.

lets compute

$$\lim_{h\to 0^+} \frac{|0+h|-|0|}{h} = \lim_{h\to 0^+} \frac{|h|}{h} = \lim_{h\to 0} \frac{h}{h} = 1$$

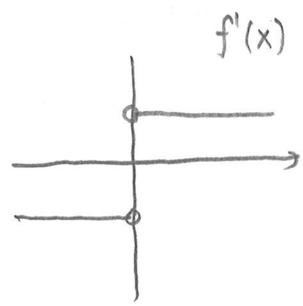
$$\lim_{h\to 0^-} \frac{|0+h|-|0|}{h} = \lim_{h\to 0^-} \frac{|h|}{h} = \lim_{h\to 0} \frac{-h}{h} = -1$$

Since these limits are different, $f'(0) = \lim\limits_{h\to 0} \dfrac{f(0+h)-f(0)}{h}$ DNE and

hence f is not differentiable at $x = 0$.

The formula for $f'(x)$ is

$$f'(x) = \begin{cases} -1, & x < 0 \\ 1, & x > 0 \end{cases}$$



The fact that $f(x) = |x|$ is not differentiable can be seen from the fact that f does not have a tangent line at $x = 0$.

Thm

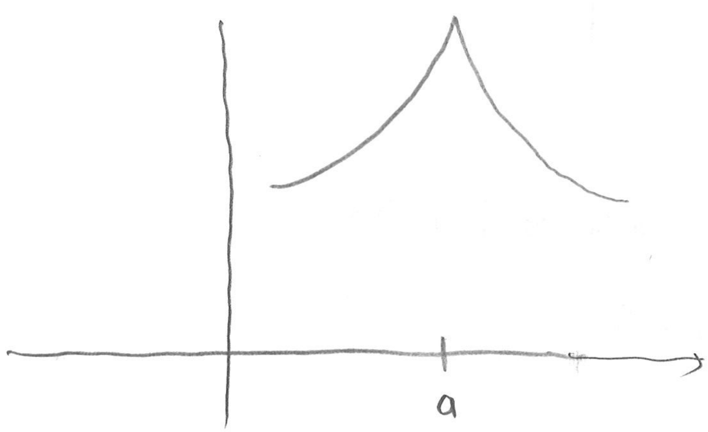
If f is differentiable at a, then f is continuous at a.

So we also have that f is not continuous at a, then f is not differentiable at a.

HOW CAN A FUNCTION FAIL TO BE DIFFERENTIABLE

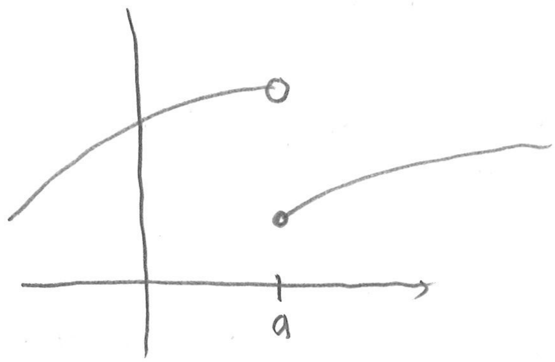
- So we saw earlier that $y = f(x) = |x|$ is not differentiable at 0, since the graph changes direction abruptly when $x = 0$.

1) In general, if the graph of a function f has a "corner" or a "Kink" in it, then the graph of f has no tangent at this point and f is not differentiable.



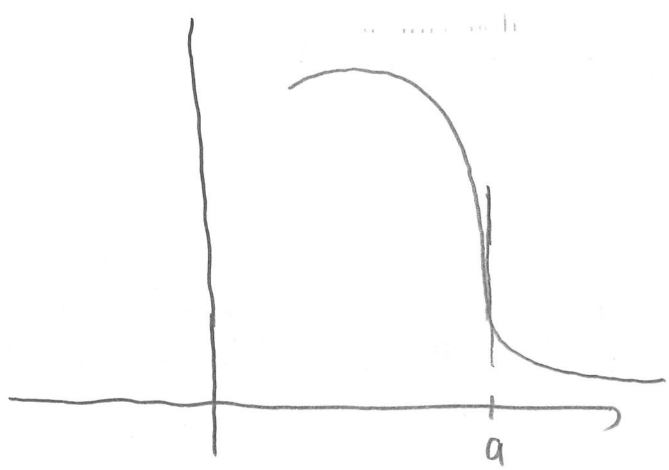
A corner

2) If f is not continuous at a, then f is not differentiable at a.



discontinuity

3) A third possibility is that the curve has a vertical tangent line when $x = a$, that is the tangent lines become steeper and steeper as $x \longrightarrow a$.



A vertical tangent

Higher Derivatives

If f is a differentiable function, then its derivative f' is also a function.

So f' may have a derivative of its own, denoted by $(f')' = f''$.

This new function f'' is called the second derivative of f

Using Leibniz notation, we can write the second derivative of $y = f(x)$ as

$$\frac{d}{dx}\left(\frac{dy}{dx}\right) = \frac{d^2y}{dx^2}$$

Ex If $f(x) = x^3 - x$, find $f''(x)$.

Earlier, we showed that the first derivative is $f'(x) = 3x^2 - 1$

Then the second derivative is :

$$f''(x) = (f'(x))' = \lim_{h\to 0} \frac{f'(x+h) - f'(x)}{h} = \lim_{h\to 0} \frac{[3(x+h)^2 - 1] - [3x^2 - 1]}{h}$$

$$= \lim_{h\to 0} \frac{3x^2 + 6xh + 3h^2 - 1 - \cancel{3x^2} + 1}{h} = \lim_{h\to 0} \frac{6xh + 3h^2}{h} = \lim_{h\to 0} \frac{h(6x+3h)}{h}$$

$$= \lim_{h\to 0} 6x + 3h = 6x$$

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Rmk $f''(x)$ can be thought of as the slope of $y = f'(x)$ at the point $(x, f'(x))$.

Acceleration

If $s = s(t)$ is the position function of an object in a straight line, we know that it's first derivative represents the velocity $v(t)$ of an object as a function of time:

$$v(t) = s'(t) = \frac{ds}{dt}$$

The instantaneous rate of change of velocity with respect to the time is called the acceleration $a(t)$ of the object.

Thus acceleration function $a(t)$ is then the derivative of the velocity function and is therefore the second derivative of the position function i.e. $a(t) = v'(t) = s''(t)$

$$\begin{array}{ccc} v'(t) & & s''(t) \\ \| & & \| \\ \dfrac{dv}{dt} & & \dfrac{d}{dt}\left(\dfrac{ds}{dt}\right) = \dfrac{d^2s}{dt^2} \end{array}$$

$\boxtimes$